Observability of the bulk Casimir effect: Can the dynamical Casimir effect be relevant to sonoluminescence?

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(Received 14 July 1997)

The experimental observation of intense light emission by acoustically driven, periodically collapsing bubbles of air in water (sonoluminescence) has yet to receive an adequate explanation. One of the most intriguing ideas is that the conversion of acoustic energy into photons occurs quantum mechanically, through a dynamical version of the Casimir effect. We have argued elsewhere that in the adiabatic approximation, which should be reliable here, Casimir or zero-point energies cannot possibly be large enough to be relevant. (About 10 MeV of energy is released per collapse.) However, there are sufficient subtleties involved that others have come to opposite conclusions. In particular, it has been suggested that bulk energy, that is, simply the naive sum of $\frac{1}{2}\hbar\omega$, which is proportional to the volume, could be relevant. We show that this cannot be the case, based on general principles as well as specific calculations. In the process we further illuminate some of the divergence difficulties that plague Casimir calculations, with an example relevant to the bag model of hadrons. [S1063-651X(98)02705-6]

PACS number(s): 78.60.Mq, 42.50.Lc, 12.20.Ds, 03.70.+k

I. INTRODUCTION

One of the most intriguing phenomena in physics today is sonoluminescence [1–3]. In the experiment, a small (radius $\sim 10^{-3}$ cm) bubble of air or other gas is injected into water, and subjected to an intense acoustic field (overpressure ~ 1 atm, frequency $\sim 2\times 10^4$ Hz). If the parameters are carefully chosen, the repetitively collapsing bubble emits an intense flash of light at minimum radius (something like a million optical photons are emitted per flash), yet the process is sufficiently noncatastrophic that a single bubble may continue to undergo collapse and emission 20 000 times a second for many minutes, if not months. Many curious properties have been observed, such as sensitivity to small impurities, strong temperature dependence, necessity of small amounts of noble gases, possible strong isotope effect, etc.

No convincing theoretical explanation of the light-emission process has yet been put forward. This is certainly not for want of interesting theoretical ideas [4]. One of the most intriguing suggestions was put forward by Schwinger [5], based on a reanalysis of the Casimir effect [6]. Specifically, he proposed that the Casimir effect, first considered by Casimir as the force between parallel conducting plates due to zero-point fluctuations in the fields [7,8], be generalized to the spherical volume defined by the bubble [9–11], and with the static boundary conditions appropriately removed. He called this idea the dynamical Casimir effect. Unfortunately, although Schwinger began the general reformulation of the static problem in Ref. [6] (most of which had been, unbeknownst to him, given earlier [12]), he did not live to complete the program. Instead, he proposed a rather naive approximation of subtracting the zero-point energy $\frac{1}{2}\Sigma\hbar\omega$ of the medium from that of the vacuum, leading, for a spherical bubble of radius a in a medium with index of refraction n, to a Casimir energy proportional to the volume of the bubble

$$E_{\rm bulk}=\frac{4\pi a^3}{3}\int\frac{(d\mathbf{k})}{(2\pi)^3}\,\frac{1}{2}k\left(1-\frac{1}{n}\right). \tag{1.1}$$

Of course, this is quartically divergent. If one puts in a suitable ultraviolet cutoff, one can indeed obtain the needed 10 MeV per flash. On the other hand, one might have serious reservations about the physical meaning of such a divergent result.

In an earlier paper, we reconsidered the Casimir effect explanation of sonoluminescence [13,14]. We argued there that the leading term (1.1) was to be removed by subtracting the contribution the formalism would make if either medium filled all space. Doing so still left us with a cubically divergent Casimir energy; but we argued further that this cubic divergence could plausibly be removed as a contribution to the surface energy. The remaining finite energy, in the presumably accurate uniform asymptotic approximation,

$$E_c\sim -\frac{(n-1)^2}{64a}, \tag{1.2}$$

is at least ten orders of magnitude too small to be relevant to sonoluminescence.

The reader might object at once that all this is in the static approximation, and the rapidly collapsing bubbles involved in sonoluminescence are anything but static. However, the time scales seem favorable for a simple adiabatic approximation to be accurate. Optical photons correspond to a time scale $\sim 10^{-15}$ s, while the flash duration is $\sim 10^{-11}$ s. That

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is, the bubble changes very little during one period of the light emitted. Of course, there may be processes here occurring on much smaller time scales, so it would be highly desirable to remove this adiabatic approximation, which we hope to accomplish in a subsequent publication.

Eberlein [15] also proposed a version of the dynamical Casimir mechanism (perhaps more properly called the Unruh [16] mechanism) as an explanation of the observed radiation. We have noted [13] technical difficulties associated with her work, especially those related to the use of ultrarelativistic velocities. See also Ref. [17]. If, in fact, reasonable numbers are used in her result, the energies involved are too small by 18 orders of magnitude, and even if her ultrarelativistic velocities are used, only 10^{-3} MeV is available. So, qualitatively, her results are not inconsistent with ours.

However, recently there has been a proposal that, indeed, the bulk energy result of Schwinger is relevant (his result, of course, is correct) [18]. These authors make an issue of the subtraction of the uniform medium contribution, implying, it would seem, that we were in error. Since this is a serious issue with experimental consequences, and since, admittedly, there are subtle issues of principle involved here, in this paper we wish to return to this point and provide further evidence for our result (1.2). In Sec. II, we will explain more fully why this subtraction was made, indicate that it has a rather long history in Casimir effect calculations, and was in fact made by Schwinger in [6] before he abandoned that effort. Then, in Sec. III, we recall the old connection between the Casimir effect and van der Waals forces, and show, in fact, that a finite energy of the same magnitude as the Casimir energy (1.2) can be obtained from the latter. Finally, motivated by recent work on regulating Casimir energies by continuing in the number of space dimensions [19], we examine, in the Appendix, whether dimensional continuation can be used to give an unambiguously finite value for the Casimir energy for a bubble in a dielectric, for example. The negative answer to the latter question shows that the quartic and cubic divergences found there are real. Again, appropriate *physical* arguments must be used to show that they are not *relevant* to the situation at hand.

II. DEFINITION OF THE CASIMIR ENERGY

In Ref. [13], we derived a formula for the Casimir energy due to electromagnetic field fluctuations in a space divided into two parts by a spherical surface of radius a. The interior region $r<a$, the inside of the bubble, has permittivity ϵ' and permeability μ', while the exterior region $r>a$, the outside of the bubble, has permittivity ϵ and permeability μ. We initially ignore dispersion. (Although it can be included [20], dispersion turns out not to affect our conclusions [13].) We calculate vacuum expectation values of field products in terms of Green's dyadics for the corresponding classical electrodynamics problem

$$i\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \mathbf{\Gamma}(\mathbf{r},\mathbf{r}'), \tag{2.1a}$$

$$i\langle \mathbf{B}(\mathbf{r})\mathbf{B}(\mathbf{r}')\rangle = -\frac{1}{\omega^2}\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\times\overleftarrow{\nabla}', \tag{2.1b}$$

where $\mathbf{\Gamma}$ is the Green's dyadic for Maxwell's equations [21]. The result for the Casimir energy is

$$E = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\ \ e^{iy\delta}\sum_{l=1}^{\infty}(2l+1)x\frac{d}{dx}\ln S_l, \tag{2.2}$$

where

$$S_l = [s_l(x')e_l'(x) - s_l'(x')e_l(x)]^2 - \xi^2[s_l(x')e_l'(x) + s_l'(x')e_l(x)]^2, \tag{2.3}$$

with

$$\xi = \frac{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}-1}{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}+1}, \tag{2.4}$$

which is expressed in terms of modified Bessel functions

$$s_l(x) = \sqrt{x}I_{l+1/2}(x), \tag{2.5a}$$

$$e_l(x) = \sqrt{x}K_{l+1/2}(x). \tag{2.5b}$$

The expression for the energy is regulated by the insertion of a Euclidean time-splitting parameter, $\delta=(x_4-x_4')/a$, and the variables are

$$x = |y|\sqrt{\epsilon\mu}, \quad x' = |y|\sqrt{\epsilon'\mu'}. \tag{2.6}$$

It is completely manifest that Eq. (2.2) does not have a well-defined limit when $\delta\to 0$—it is quartically divergent. Indeed, it is easy to show, as Ref. [18] does, that the quartically divergent term here corresponds precisely to the Schwinger result (1.1) when $\epsilon'=\mu'=1$ and $\mu=1$. However, it is also quite clear that the calculation is not yet done when we have reached this point. As we stated in Ref. [13], ''We must remove the term which would be present if either medium filled all space (the same was done in the case of parallel dielectrics Ref. [21]).'' When we look at the latter reference, we see immediately the point. Again to quote, this time from Ref. [21]: ''These terms [to be subtracted] correspond to the electromagnetic energy required to replace medium 1 by medium 2 in the displacement volume. (Since this term in the energy is already phenomenologically described, it must be cancelled by an appropriate contact term.)'' What we were saying there, in the present context, is that the term in the energy corresponding to the boundary-condition-independent Green's function

$$F_l^{(0)} = ikj_l(kr_<)h_l^{(1)}(kr_>), \tag{2.7}$$

must be removed, because it contributes (a formally infinite amount) to the bulk energy of the material, which is already phenomenologically described in terms of its bulk properties. In fact, we are not creating material, e.g., water, we are simply displacing it when we insert the bubble, and force the bubble to expand and contract. The energy per unit element of medium is therefore not changed. (The density of the air in the bubble of course changes greatly, but the zero-point

energy of that relatively dilute medium is certainly insignificant because $n \approx 1$. In any case, the effect of this density change is also included in the phenomenological description.) Indeed, the spectacular agreement between the the Lifshitz theory of parallel dielectrics [22], rederived in Ref. [21], and the experiment of Sabisky and Anderson [23] seems strong vindication of this subtraction procedure.

Further evidence that we are on the right track is provided by Schwinger himself. In the third article cited in Ref. [6], where he rederived the result for parallel dielectrics, he explicitly removed volume and surface energies: ''... one finds contributions to E that, for example, are proportional ... to the volume enclosed between the slabs. The implied constant energy density—independent of the separation of the slabs—violates the normalization of the vacuum energy density to zero. Accordingly, the additive constant has a piece that maintains the vacuum energy normalization. There is also a contribution to E that is proportional to [the area], energy associated with individual slabs. The normalization to zero of the energy for an isolated slab is maintained by another part of the additive constant.'' Admittedly, the situation is more clear cut in the parallel-plate geometry. However, in the following paper (the last reference in Ref. [6]), where Schwinger began to set up the problem for the spherical geometry (but left the details to Harold [24]), a close reading shows that a similar subtraction is implicit. Unfortunately, when Schwinger went on to apply Casimir energy to sonoluminescence in Ref. [5], he did not make use of the general analysis in Ref. [6]. Instead, needing an immediate result to confront the phenomenology, Schwinger simply jumped to the unsubtracted, unregulated result (1.1)—see the second reference in Ref. [5].

But enough of argumentation. Let us turn to detailed calculations that support our contention.

III. DERIVATION OF CASIMIR EFFECT FROM VAN DER WAALS FORCES

It is familiar that the van der Waals forces between polarizable molecules—the Casimir-Polder forces [25]—can be derived from the Casimir forces between dielectric bodies. We interpret this as meaning that the Casimir effect is merely a local field form of the action-at-a-distance summation of the forces between the molecules that make up the material bodies.

Let us begin with a variation of the argument given in Ref. [21]. Consider a dielectric slab bounded by planes $z=0$ and $z=a$, having a dielectric constant ϵ; outside this region there is vacuum, $\epsilon=1$. According to the Lifshitz formula [21,22], the force per area between the surfaces is

$$f=-\int_0^\infty \frac{d\zeta}{2\pi}\int_0^\infty \frac{dk^2}{2\pi}\kappa_3\left\{\left[\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)^2 e^{2\kappa_3 a}-1\right]^{-1} + \left[\left(\frac{\kappa_3'+\kappa_1'}{\kappa_3'-\kappa_1'}\right)^2 e^{2\kappa_3 a}-1\right]^{-1}\right\}, \tag{3.1}$$

where, in the ith medium (we denote the region of the slab by 3, that the outside regions by 1),

$$\kappa_i^2=k^2+\epsilon_i\zeta^2, \quad \kappa_i'=\frac{\kappa}{\epsilon_i}. \tag{3.2}$$

Now suppose the medium is tenuous, so that the dielectric constant differs only slightly from unity,

$$\epsilon-1\ll 1. \tag{3.3}$$

Then, with a simple change of variable,

$$\kappa=\zeta p, \tag{3.4}$$

we can recast the Lifshitz formula (3.1) into the form

$$f\approx -\frac{1}{32\pi^2}\int_0^\infty d\zeta\ \zeta^3\int_1^\infty \frac{dp}{p^2}[\epsilon(\zeta)-1]^2 \times[(2p^2-1)^2+1]e^{-2\zeta p a}. \tag{3.5}$$

If the separation of the surfaces is large compared to the characteristic wavelength characterizing ϵ, $a\zeta_c\gg 1$, we can disregard the frequency dependence of the dielectric constant, and we find

$$f\approx -\frac{23(\epsilon-1)^2}{640\pi^2 a^4}. \tag{3.6}$$

For short distances $a\zeta_c\ll 1$, the approximation is

$$f\approx -\frac{1}{32\pi^2}\frac{1}{a^3}\int_0^\infty d\zeta[\epsilon(\zeta)-1]^2. \tag{3.7}$$

These formulas are identical with the well-known forces found for the complementary geometry in Ref. [21].

Now we wish to derive these results from the sum of van der Waals forces, derivable from a potential of the form

$$V=-\frac{B}{r^\gamma}. \tag{3.8}$$

We do this by computing the energy (N is the density of molecules)

$$E=-\frac{1}{2}BN^2\int_0^a dz\int_0^a dz'\int (d\mathbf{r}_\perp) \times(d\mathbf{r}_\perp')\frac{1}{[(\mathbf{r}_\perp-\mathbf{r}_\perp')^2+(z-z')^2]^{\gamma/2}}. \tag{3.9}$$

If we disregard the infinite self-interaction terms (see below), we obtain

$$f=-\frac{\partial}{\partial a}\frac{E}{A}=-\frac{2\pi BN^2}{(2-\gamma)(3-\gamma)}\frac{1}{a^{\gamma-3}}. \tag{3.10}$$

So then, upon comparison with Eq. (3.6), we set $\gamma=7$, and, in terms of the polarizability,

$$\alpha=\frac{\epsilon-1}{4\pi N}, \tag{3.11}$$

we find

$$B=\frac{23}{4\pi}\alpha^2, \tag{3.12}$$

or, equivalently, we recover the retarded dispersion potential

$$V=-\frac{23}{4\pi}\frac{\alpha^2}{r^7}, \tag{3.13}$$

whereas for short distances we recover the London potential

$$V=-\frac{3}{\pi}\frac{1}{r^6}\int_0^\infty d\zeta\ \alpha(\zeta)^2. \tag{3.14}$$

Our intention is to carry out the same simple calculation for a dielectric sphere. The first couple of steps are unambiguous (θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$):

$$E=-\frac{1}{2}BN^2\int (d\mathbf{r})(d\mathbf{r}')\frac{1}{(r^2+r'^2-2rr'\cos\theta)^{\gamma/2}}$$
$$=-\frac{4\pi^2BN^2}{2-\gamma}\int_0^a dr\int_0^a dr'rr'\left[\frac{1}{(r+r')^{\gamma-2}}-\frac{1}{|r-r'|^{\gamma-2}}\right]. \tag{3.15}$$

Now, however, there are divergences of two types, ''volume'' ($r'\to r$) and ''surface'' ($r\to a$). The former is of a universal character. If we regulate it by a naive point separation, $r'\to r+\delta$ and $\delta\to 0$, we find the most divergent part to be

$$E_{\rm vol}=-\frac{\pi BN^2}{10}\frac{1}{\delta^4}V,\quad V=\frac{4\pi a^3}{3}, \tag{3.16}$$

which is identical to the corresponding (omitted) divergent term in the parallel dielectric calculation, where $V=aA$. This is obviously the self-energy divergence that would be present if the medium filled all space, and makes no reference to the interface, and is therefore quite unobservable. This is the analog (although the ϵ dependence is different) of the volume divergence in the Casimir effect [Eq. (1.1)].

If, once again, the divergent terms are simply omitted, as may be weakly justified by continuing in the exponent γ from $\gamma<3$, we obtain a positive energy,

$$E_{\rm vdW}=\frac{23}{1536\pi a}(\epsilon-1)^2. \tag{3.17}$$

This may be more rigorously justified by continuing in dimension, a procedure which has proved useful and illuminating in Casimir calculations [19]. Thus we replace the previous expression for the energy by

$$E=-\frac{1}{2}BN^2\int d^Dr\ d^Dr'\frac{1}{|\mathbf{r}-\mathbf{r}'|^\gamma}, \tag{3.18}$$

where, in terms of the last angle in D-dimensional polar coordinates,

$$\int d^Dr=\frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}\int_0^a dr\ r^{D-1}\int_0^\pi d\theta\ \sin^{D-2}\theta. \tag{3.19}$$

If we take, say, $\mathbf{r}'$ to lie along the z axis, so that θ is again the angle between $\mathbf{r}$ and $\mathbf{r}'$, we find

$$E=-\frac{1}{2}BN^2\frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}\int_0^a dr'\ r'^{D-1}\int_0^a dr\ r^{D-1}$$
$$\times\int_{-1}^1 d\cos\theta(1-\cos^2\theta)^{(D-3)/2}$$
$$\times(r^2+r'^2-2rr'\cos\theta)^{-\gamma/2}. \tag{3.20}$$

The angular integration can be given in terms of an associated Legrendre function $P_b^a(z)$,

$$\int_{-1}^1 dt(1-t^2)^{(D-3)/2}(r^2+r'^2-2rr't)^{-\gamma/2}=\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)$$
$$\times(rr')^{1-D/2}|r^2-r'^2|^{(D-\gamma-2)/2}P^{1-D/2}_{(\gamma-D)/2}\left(\frac{r^2+r'^2}{|r^2-r'^2|}\right). \tag{3.21}$$

Now let us substitute this into the expression for the energy, and change variables from r, r' to

$$x=r^2+r'^2,\quad y=\frac{r^2+r'^2}{|r^2-r'^2|}. \tag{3.22}$$

The x integral is then trivially done, leaving us with

$$E=-\frac{BN^2\pi^D}{2^{D/2}\Gamma(D/2)}\frac{1}{D-\gamma/2}\int_1^\infty dy\left(\frac{2a^2}{y+1}\right)^{D-\gamma/2}$$
$$\times(y^2-1)^{(D-2)/4}P^{1-D/2}_{(\gamma-D)/2}(y), \tag{3.23}$$

valid for $D>\gamma/2$. Integrals of this type are given in [26]:

$$\int_1^\infty dy(y-1)^{-a/2}(y+1)^{b+a/2-1}P_b^a(y)$$
$$=2^b\frac{\Gamma(-2b)}{\Gamma(1-b-a)\Gamma(1-b)}, \tag{3.24}$$

valid for $\mathrm{Re}\,a<1$, $\mathrm{Re}\,b<0$. Then we have, using the duplication formula for the Γ function,

$$E=-BN^2\frac{\pi^{D-1/2}2^{D-\gamma}\Gamma\left(\frac{D-\gamma+1}{2}\right)}{\Gamma(D/2)\Gamma(D-\gamma/2+1)(D-\gamma)}. \tag{3.25}$$

The resulting formula is regular when D and γ are both odd integers, so we can analytically continue from $D>\gamma$ to $D=3$ for $\gamma=7$. Doing so gives us, using Eq. (3.12),

$$E = BN^2 \frac{\pi^2}{24}\frac{1}{a} = \frac{23}{24}\frac{(\epsilon-1)^2}{64\pi a}, \tag{3.26}$$

exactly the same as Eq. (3.17). Note that the magnitude of this result is nearly the same as that found in Ref. [13], and stated in Eq. (1.2), differing only by the factor

$$\frac{23}{24}\frac{4}{\pi} = 1.22, \tag{3.27}$$

which is a plausible difference in that the previous calculation was only in the leading asymptotic approximation, but the sign is opposite. We offer as evidence for the validity of this methodology the fact that formula (3.25) gives the correct Coulomb energy of a uniform ball of charge, for which $\gamma = 1$.

Evidently, we have reached the frontier of our understanding of the Casimir effect and its connection with van der Waals forces. The subtraction procedure may well be ambiguous, although the volume and surface divergences are unambiguous. That these divergences are real is further reinforced by the considerations of the Appendix, which shows that the technique of dimensional continuation fails for this case. But these divergences are not *relevant* to the light emission process, although they would be to a first-principles calculation of the energy density and surface tension of the medium [21]. However, our qualitative conclusion, that quantum vacuum energies are completely irrelevant to sonoluminescence, is dependent only on the order of magnitude of the finite remainder, given by either Eq. (1.2) or (3.17).

IV. CONCLUSIONS

Our conclusions here are threefold.

(i) The divergences that occur when interior and exterior modes are mismatched, whether by exclusion of one set, or by changing the speed of light in the two media, are real, and cannot be circumvented by a mathematical trick.

(ii) Volume divergences are not physically meaningful, since they reflect self-energy effects, and serve to define the intrinsic properties of the material. They are naturally canceled out by the introduction of a suitable contact term. What is left is a surface divergence, which presumably is physically meaningful, yet should be absorbed into a renormalization of physical parameters, such as the surface tension.

(iii) The magnitude of the finite remainder, of order $1/a$, apparently may be extracted unambiguously. Whatever its sign, it is far too small to be relevant to sonoluminescence.

ACKNOWLEDGMENTS

We thank M. Visser for a discussion about his work and the significance of the bulk energy contribution. We thank the U.S. Department of Energy for their financial support of this work.

APPENDIX: DIMENSIONAL CONTINUATION OF THE CASIMIR EFFECT

The fact that the above dimensional regularization of the van der Waals energy gave a finite result suggests that we re-examine the Casimir calculation to see if possibly an unambiguous finite result could thereby be obtained. We will not be surprised to find a negative answer to this question, since the perfect cancellation between interior and exterior modes cannot hold true with different speeds of light in the two media [27].

We will content ourselves by examining the extreme case of $\epsilon \to \infty$ in the exterior region, that is, a bag with perfectly conducting boundary conditions on the surface. Since it is necessary to continue the individual modes, we will examine the TE mode as representative. (As we will see, the subleading divergences cancel between the TE and TM modes.) In three dimensions the interior modes alone give [28]

$$E_{\rm in}^{\rm TE} = -\frac{1}{2\pi a}\sum_{n=1}^{\infty}(2n+1)\int_0^\infty dx\, x\frac{s_n'(x)}{s_n(x)}, \tag{A1}$$

where the generalized modified Ricatti-Bessel functions are

$$s_n(x) = x^{D/2-1}I_\nu(x), \quad e_n(x) = x^{D/2-1}K_\nu(x), \tag{A2}$$

where $\nu = n + D/2 - 1$ ($= n + 1/2$ here). The generalization of this result to D space dimensions is [19]

$$E_{\rm in}^{\rm TE} = -\frac{1}{2\pi a}\frac{1}{\Gamma(D-1)}\sum_{n=1}^{\infty} w(n,D)\int_0^\infty dx\, x\frac{s_n'(x)}{s_n(x)}, \tag{A3}$$

where the weight function is

$$w(n,D) = \frac{2\nu\Gamma(n+D-2)}{n!}. \tag{A4}$$

Again, as elsewhere, in Ref. [28] the ''vacuum energy'' term was subtracted. (As noted in the text, the justification was only partly that it removed the most divergent terms.) This was obtained from the free Green's function, which in D space dimensions is

$$\mathcal{G}_\omega^0(r,r',\theta) = i\sum_n \frac{2\nu\Gamma\left(\frac{D}{2}-1\right)}{8(\pi r r')^{D/2-1}} C_n^{(D/2-1)} \times (\cos\theta) J_\nu(kr_<) H_\nu^{(1)}(kr_>), \tag{A5}$$

in terms of the ultraspherical or Gegenbauer polynomial. The stress on the sphere is obtained by applying the appropriate differential operator corresponding to the stress tensor,

$$T_{rr} = \frac{i}{2}(\nabla_r\nabla_{r'} + \omega^2 - \nabla_\perp\cdot\nabla_{\perp'})\mathcal{G}_\omega^0 \to \frac{i}{2}\left(r^{D-2}\partial_r r^{2-D} r'^{D-2}\partial_{r'} r'^{2-D} + \omega^2 - \frac{n(n+D-2)}{r^2}\right)\mathcal{G}_\omega^0. \tag{A6}$$

Subtracting this from the previous result gives

$$E_{\rm in}^{\rm TE}=-\frac{1}{2\pi a}\frac{1}{\Gamma(D-1)}\sum_{n=1}^{\infty}w(n,D)\times\int_0^\infty dx\;x\left\{\frac{s_n'(x)}{s_n(x)}+x^{3-D}\left[s_n'(x)e_n'(x)-\left(1+\frac{n(n+D-2)}{x^2}\right)s_n(x)e_n(x)\right]\right\}. \quad \text{(A7)}$$

The question now is whether the continuation procedure described in Ref. [19] can be successfully applied here. There, we first made the integrals convergent by adding a suitable term to the summand which sums to zero for sufficiently small dimension. Here this suggests that in the above integral we replace

$$\frac{s_n'(x)}{s_n(x)}=\frac{d}{dx}\ln x^{D/2-1}I_\nu(x)\to\frac{d}{dx}\ln\sqrt{2\pi x}I_\nu(x), \quad \text{(A8)}$$

for then the large x behavior of this term is $1+(4\nu^2-1)/(8x^2)+\cdots$. The vacuum subtraction term cancels the leading term here, leaving for the leading term in the braces in Eq. (A7):

$$\frac{(D-3)(D-2)}{4x^2}+O(x^{-4}). \quad \text{(A9)}$$

So, not surprisingly, the integral is still, in general, logarithmically divergent, *although for $D=3$ or 2 it does converge.*

Therefore, it appears that we cannot meaningfully continue off the integers. So we are forced to retreat back to $D=3$. There we have

$$E_{\rm in}^{D=3,\rm TE}=-\frac{1}{2\pi a}\sum_{n=1}^{\infty}(2n+1)Q_n, \quad \text{(A10)}$$

where Q_n is the convergent integral,

$$Q_n=\int_0^\infty dx\;x\left\{\frac{d}{dx}\ln\sqrt{2\pi x}I_\nu(x)+\text{c.t.}\right\}. \quad \text{(A11)}$$

If we use the uniform asymptotic expansions for the Bessel functions, we easily find

$$Q_n\sim\nu^2\int_0^\infty dz\;z\left[\frac{t^2}{8\nu z}-\frac{t^3(2t^2+3)}{4\nu^2 z}\right]\sim\frac{\nu\pi}{16}+O(\nu^0), \quad (\nu\to\infty), \quad \text{(A12)}$$

where $t=(1+z^2)^{-1/2}$. The leading term here is precisely the negative of that found in the TM mode; that is, for electrodynamics, or linearized QCD, the interior modes contribute a quadratically divergent sum, rather than the cubically divergent one due to each mode. Practical methods of dealing with this divergent Casimir energy, which is relevant in hadronic physics, were suggested in Ref. [29].

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